Lec 11: A short note on production with several inputs and isoquants

Production of the good produced, Q, takes place with two inputs, K (man made capital) and R (natural resources) with the respective input prices δ (the interest rate for K) and v (the cost of acquiring R). This note uses N in stead of R for **N**atural resources.

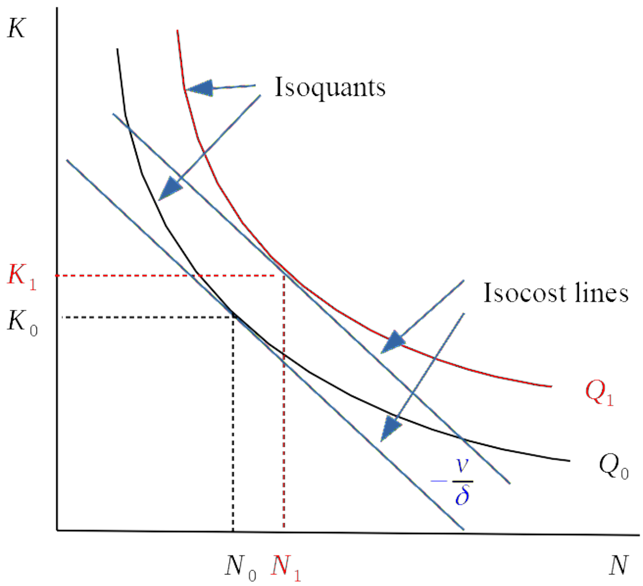
The text book's mathematical formulation: $Q=Q(K,R)$. This note: $Q=Q(K,N)$

This gives the profit function: $\pi = \underbrace{pQ(K,N)}_{\text{Revenues}} - \underbrace{(\delta K + vN)}_{\text{Costs}}$ with K and N as choice variables.
First order conditions for profit maximization:

FOC profit max: (a) $\frac{\partial \pi}{\partial K} = pQ_K(K,N) - \delta = 0$ and (b) $\frac{\partial \pi}{\partial N} = pQ_N(K,N) - v = 0$ which gives the profit maximizing solution $\{K^*, N^*\}$ with the interpretations that marginal revenues for each input factor equals its marginal costs (input prices).

Along an isoquant production is kept constant, i.e., $Q = Q_0$.

Graphically (for illustrative purposes two isoquants are depicted: $Q_0 < Q_1$):



Isocost lines

- = equal cost line: costs the same along the line
- Slope determined by the relative price (here $-v/\delta$)
- Lower cost the closer to the origo

On the optimal solution:

- Within an isoquant-isocost framework the optimal (cost min) solution is given by the tangency between the isoquant and the isocost line
  - for isoquant $Q_0 : \{K_0, N_0\}$
  - for isoquant $Q_1 : \{K_1, N_1\}$
  - Remark: to produce more, one needs to use more of the inputs K and N

Mathematically: $\min_{\{K,N\}} \delta K + vN$ subject to $Q(K,N) \geq Q_0$

which gives the Lagrangian: $\mathcal{L}(K,N,\lambda) = \delta K + vN + \lambda(Q_0 - Q(K,N))$

First order conditions by differentiating with the choice variables (K and N plust the lagrangian multiplier = shadow price λ). Note that we avoid the Kuhn-Tucker conditions ($\geq$ for a minimization problem like here, or $\leq$ for maximization problem) using the equality constraint $Q(K,N) \geq Q_0$, which solves the cost min for this problem:

(1) $\frac{\partial \mathcal{L}}{\partial K} = \delta - \lambda Q_K(K,N) = 0 \Rightarrow \lambda Q_K(K,N) - \delta = 0$

(2) $\frac{\partial \mathcal{L}}{\partial N} = v - \lambda Q_N(K,N) = 0 \Rightarrow \lambda Q_N(K,N) - v = 0$

(3) $\frac{\partial \mathcal{L}}{\partial \lambda} = Q_0 - Q(K,N) = 0$

with the solution $\{K^{cm}, N^{cm}, \lambda\}$.

If the solution of the unconstrained profit maximization problem $\pi = p\,Q(K,N) - (\delta K + v N)$ and the constrained cost. min problem $\mathscr{L}(K,N,\lambda) = \delta K + v N + \lambda(Q_0 - Q(K,N))$ coincide, i.e., $K^{cm} = K^*, N^{cm} = N^*$, then $Q(K^*, N^*) = Q_0$ and the lost profits are 0. The shadow price (here λ) is therefore also zero.

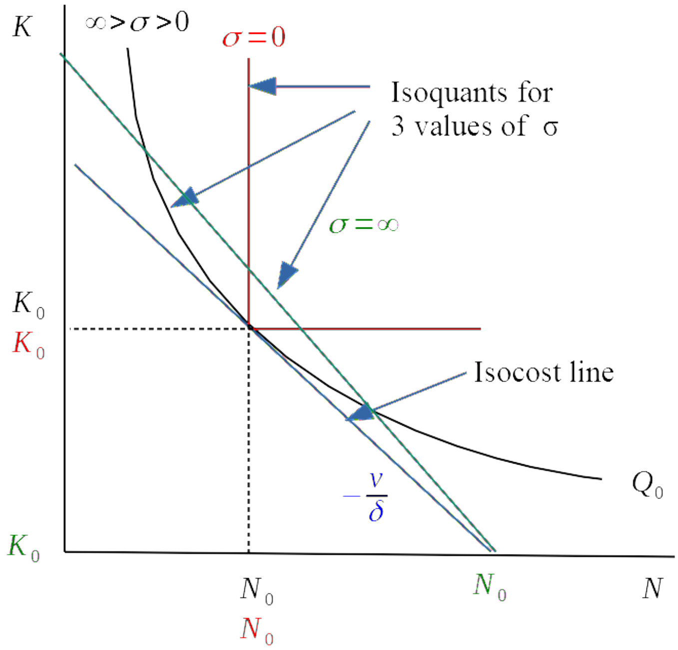
The shadow price measures the marginal change in the objective value between the optimal (unconstrained) solution and the constrained solution in question. In this case:

- $Q(K^*, N^*) < Q_0 \Rightarrow \lambda < 0 = -p$ as the equality constraint forces the excess production level.
- $Q(K^*, N^*) > Q_0 \Rightarrow \lambda > 0 = p$ as the eequality constraint prevents reaching the profit maximizing production level.

The elasticity of substitution and its impacts on input factor use

Definition of elasticity of substitution: $\sigma = \dfrac{\dfrac{d(N/K)}{N/K}}{\dfrac{d(Q_K/Q_N)}{Q_K/Q_N}}$

It measures how input use (here: man made capital *K* and natural capital *N*) changes along an isoquant as percent change in input use over percent use in marginal productivity. This measure is relevant for the strong and weak sustainability debate (lower elasticity of substitution reduces the likelihood for weak sustainability to hold).



Three cases for σ:

- $\sigma = 0$: Fixed proportions (Leontief production). The optimal input mix is completely insensitive to changes in relative prices (v/δ).
- $\infty > \sigma > 0$: Diminishing marginal input factor productivity (most common case). Optimal input mix changes when relative prices (v/δ) such that less is used of the input that becomes more expensive.
- $\sigma = \infty$: $Q = a\,K + b\,N$ (linear production) which gives a straight line isoquant. Optimal input mix: one input is zero and the other input is where the isoquant crosses the axis ($K_0 = 0$ and N_0 in graph) depending upon the steepness of relative prices (v/δ) compared to the steepness of b/a the production function coefficients.

This implies that changes in relative input prices (v/δ) will influence how easy it is for the economy to adjust/the cost of adjustment in the production. To see this, consider $\sigma = \infty$ where an increase in the price of one input may leave the input mix unchanged as long as the slope of v/δ remains steeper/-flatter than b/a before the price change in the example above.