Lecture 11 (add): Elasticity of substitution for the Cobb-Douglas production function

Recall the definition the *elasticity of substitution*: $\sigma = \dfrac{\frac{d(R/K)}{R/K}}{\frac{d(Q_K/Q_R)}{Q_K/Q_R}} = \dfrac{d(R/K)}{R/K}\dfrac{Q_K/Q_R}{d(Q_K/Q_R)}$

where $d(\ldots)$ is the total differential of the term in the bracket

The CD production function: $Q = AK^{\alpha}R^{\beta}, A>0, \alpha>0, \beta>0$
(Remark: $\alpha+\beta=1$ is constant returns to scale, $\alpha+\beta<1$ is decreasing returns to scale)

with first order derivatives: $Q_K = \dfrac{\partial Q}{\partial K} = \alpha A K^{\alpha-1}R^{\beta}$ and $Q_R = \dfrac{\partial Q}{\partial R} = \beta A K^{\alpha}R^{\beta-1}$ which

gives the marginal rate of technical substitution $MRTS_{KR} = Q_K/Q_R = \dfrac{\alpha A K^{\alpha-1}R^{\beta}}{\beta A K^{\alpha}R^{\beta-1}} = \dfrac{\alpha}{\beta}\dfrac{R}{K}$ (1)

The total derivative (= differentiation with all choice variables in a function) of *R/K*:

$$d(R/K) = \left(\frac{-R}{K^2}dK + \frac{K}{K^2}dR\right) = (-R\,dK + K\,dR)\frac{1}{K^2} \quad (2)$$

Above derivative obtained from the quotient rule https://en.wikipedia.org/wiki/Quotient_rule

Along the isoquant the change in R (dR) in terms of the change in K (dK) must equal:

$$dR = -Q_K/Q_R\,dK = \frac{\alpha}{\beta}\frac{R}{K}dK \quad \text{(where we use (1))}$$

Insert above line into (2) to get:

$$d(R/K) = (-R\,dK + K\,dR)\frac{1}{K^2} = \left(-R\,dK + K\frac{\alpha}{\beta}\frac{R}{K}dK\right)\frac{1}{K^2} \quad \text{(2) and (3)}$$

Next, we need to evaluate $d(Q_K/Q_R)$

As $Q_K/Q_R = \dfrac{\alpha}{\beta}\dfrac{R}{K}$ we can use (2) and write

$$d(Q_K/Q_R) = \frac{\alpha}{\beta}d(R/K) = \frac{\alpha}{\beta}\left(-R\,dK + K\frac{\alpha}{\beta}\frac{R}{K}dK\right)\frac{1}{K^2}$$

We now has all the "building blocks" needed to find the value of

$$\sigma = \frac{d(R/K)}{R/K}\frac{Q_K/Q_R}{d(Q_K/Q_R)} = \frac{d(R/K)}{R/K}\frac{Q_K/Q_R}{\frac{\alpha}{\beta}d(R/K)} = \frac{1}{\frac{R}{K}}\frac{\frac{\alpha}{\beta}\frac{R}{K}}{\frac{\alpha}{\beta}} = 1$$

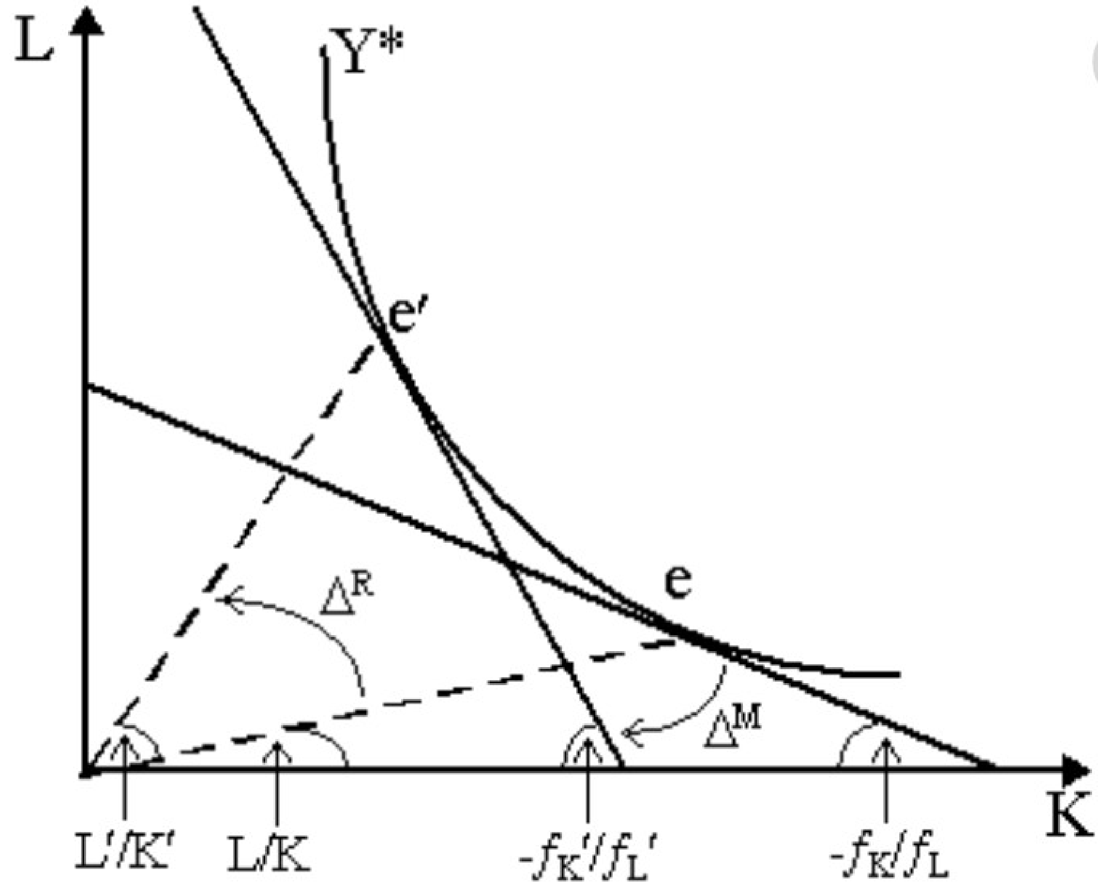
: along any isoquant for the Cobb-Douglas production function, the elasicity of substitution is **one**

Remark: there are numerous other ways of calculating the elasticity of substitutio. One of the "good" ways (with a mathematical trick) is on Wikipedia:
https://en.wikipedia.org/wiki/Elasticity_of_substitution

A brief graphical presentation of the elasticity of substitution (next page)

A brief graphical presentation of the elasticity of substitution

The figure below has replaced natural capital *R* with labor *L*, but it still captures the essentials



Remark: The notation in the graph relates to two different cost minimum allocations, *e* and *e'*, which the follows the different marginal rates of substitution used ($-f_K/f_L$ and $-f_K'/f_L'$), and the corresponding input factor ratios *L/K* and *L'/K'*.