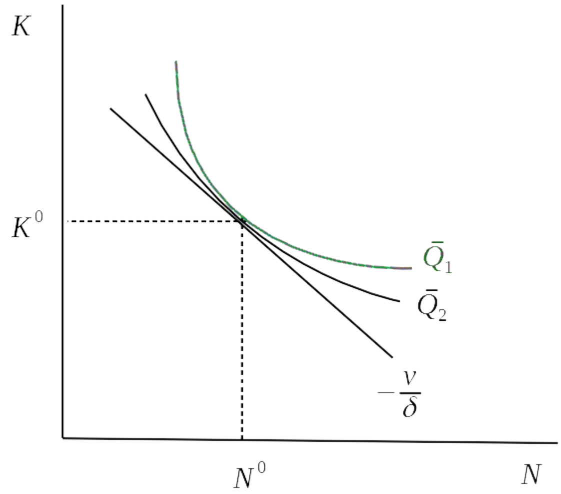
# ECN 275/375 Environmental and natural resource economics
# Exercise set 11 – Eirik's suggested answers

**Exercise 11.1 – Isoquants**

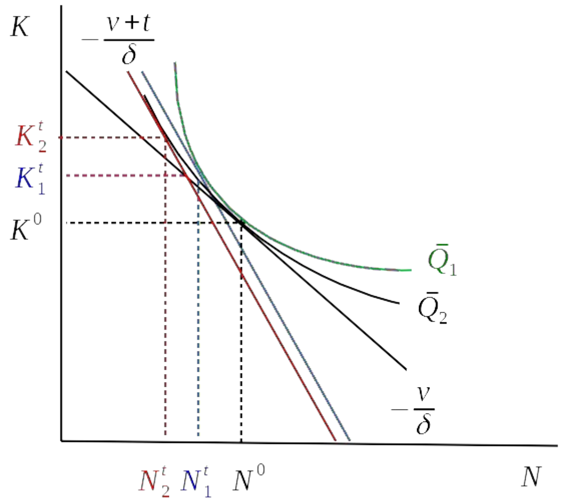
Two production functions are of the form $Q_i = Q_i(K, N)$, where $K$ is man-made capital, $N$ is natural capital, and $i \in \{1, 2\}$. Both production functions are regular with (standard) convex isoquants to the origon with , where the respective elsticities of substitution, $\sigma_1$ and $\sigma_2$, have the following values: $0 < \sigma_1 < \sigma_2 < \infty$. The input factor prices are $\delta$ and $v$ for the two input factors.

(a) Draw the two isoquants for the same production level, $Q_1^0 = Q_2^0$, and show the resulting optimal uses of the two inputs.

**Answer:** The optimal input factor use is given where the negative relative price line $-v/\delta$ tangents the isoquants. In my graph I drew the two isoquants such that the optimal input uses are the same, i.e., $K_1^0 = K_2^0 = K^0$ and $N_1^0 = N_2^0 = N^0$.




(b) Suppose a tax $t$ is added to the price of natural capital. From a producer's perspective, what is the price on the use of natural resources? Show graphically that the relative price change gives a larger input factor substitution from the price change for the production function that has the highest value for the elasticity of substitution (isoquant 2).




**Answer:** The new input factor price for natural resources is $v + t$. This gives a steeper relative price line $-(v+t)/\delta$ than in the initial case (in a), and the new allocations are $K_1^t$ and $N_1^t$ for the first production function and are $K_2^t$ and $N_2^t$ for the second production function. Note that to get the new relative price line (isocost line) to tangent the least substitution elastic isoquant (1), the budget line is located a bit further out (northeast) compared to for most substitution elastic isoquant (2).

(c) Explain technically and intuitively why the cost increase of the tax is larger for the least substitution elastic production function (isoquant 1).

**Answer:** Technically, because the relative price line (the blue isocost line) for the least substitution elastic production function is located further to the northeast than for the most substitution elastic production function (the red isocost line). Recall that the further out the isocost line is located, the higher the costs.

Intuitively, the more substitution elastic the production functin is (straighter isoquant), the higher is the possibility of substituting the input factor (here $N$) that has become more expensive.

**Exercise 11.2 – Cobb-Douglas example of unconstrained and constrained production**

The production function is: $Q(K,N)=AK^{\alpha}N^{\beta}$ with the input factor $K$ (man-made capital) and $N$ (natural capital), and with $A>0, 1>\alpha>0, 1>\beta>0$. The product price is $p > 0$, and the input factor prices are $\delta>0, v>0$ respectively for $K$ and $N$.

(a) What is the profit function?

**Answer**: $\pi(K,N)=pAK^{\alpha}N^{\beta}-\delta K-vN$

(b) Show that the input factor demand functions expressed in terms of the parameters and the prices ($p$, $\delta$ and $v$) becomes: $N(p,\delta,v)^{*}=\left(\frac{1}{pA}\right)^{\frac{1}{\alpha+\beta-1}}\left(\frac{\beta}{v}\right)^{\frac{\alpha-1}{\alpha+\beta-1}}\left(\frac{\delta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta-1}}$ and $K(p,\delta,v)^{*}=\left(\frac{1}{pA}\right)^{\frac{1}{\alpha+\beta-1}}\left(\frac{\alpha}{\delta}\right)^{\frac{\beta-1}{\alpha+\beta-1}}\left(\frac{v}{\beta}\right)^{\frac{\beta}{\alpha+\beta-1}}$. **Remark: Not exam curriculum**

**Answer:** Differentiate the profit function with respect to the choice variables $K$ and $N$ to get the first order conditions for profit maximization:

(1) $\frac{\partial\pi}{\partial K}=pA\alpha K^{\alpha-1}N^{\beta}-\delta=0$ and (2) $\frac{\partial\pi}{\partial N}=pA\beta K^{\alpha}N^{\beta-1}-v=0$

Move the input factor prices to the right hand side (RHS) and take the ratio to get

$\frac{pA\alpha K^{\alpha-1}N^{\beta}}{pA\beta K^{\alpha}N^{\beta-1}}=\frac{\delta}{v}$ where red terms cancel out and blue terms simplify so we get $\frac{\alpha}{\beta}\frac{N}{K}=\frac{\delta}{v}$.

Solve for $K$ as a function of the other input $N$ and the parameters to get: $K=\frac{\alpha}{\beta}\frac{v}{\delta}N$, which describes the expansion path for $K$ as a function of $N$ (note that for Cobb-Douglas this is linear in $N$, i.e., same ratio of inputs when production is expanded). Substitute $K$ into (2):

$pA\alpha K^{\alpha}N^{\beta-1}-v=pA\beta\left(\frac{\alpha}{\beta}\frac{v}{\delta}N\right)^{\alpha}N^{\beta-1}-v=0$, which we reorganize to get:

$N^{\alpha+\beta-1}pA\beta\left(\frac{\alpha}{\beta}\frac{v}{\delta}\right)^{\alpha}=v$ which gives:

$$N^{\alpha+\beta-1}=\frac{v}{pA\beta}\left(\frac{\alpha}{\beta}\frac{v}{\delta}\right)^{-\alpha}=\frac{v}{pA\beta}\left(\frac{\beta}{\alpha}\frac{\delta}{v}\right)^{\alpha}=\frac{1}{pA}\left(\frac{\beta}{v}\right)^{\alpha-1}\left(\frac{\delta}{\alpha}\right)^{\alpha}=\frac{1}{pA}\left(\frac{v}{\beta}\right)^{1-\alpha}\left(\frac{\delta}{\alpha}\right)^{\alpha},$$

which gives the demand function for $N$ expressed in the parameters:

$$N(p,\delta,v)^{*}=\left(\frac{1}{pA}\right)^{\frac{1}{\alpha+\beta-1}}\left(\frac{\beta}{v}\right)^{\frac{\alpha-1}{\alpha+\beta-1}}\left(\frac{\delta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta-1}}$$

The structure of the production function allows us directly to write the demand function for $K$:

$K(p,\delta,v)^{*}=\left(\frac{1}{pA}\right)^{\frac{1}{\alpha+\beta-1}}\left(\frac{\alpha}{\delta}\right)^{\frac{\beta-1}{\alpha+\beta-1}}\left(\frac{v}{\beta}\right)^{\frac{\beta}{\alpha+\beta-1}}$ (if in doubt, insert the expansion path for $N$ as a function of $K$ (= $\frac{\beta}{\alpha}\frac{\delta}{v}N$ ) and plug this into (1). Go through the same steps as done above for $N^*$ to find $K^*$.

(c) Suppose that production is quantity restricted to $\bar{Q}$ . Set up the appropriate optimization problem, the corresponding Lagrangian, and the first order conditions.

**Answer:** This is a cost minimization problem to produce $\bar{Q}$ , i.e. the least cost solution for producing $\bar{Q}$ . The cost min problem:

$$\min_{\{K,N\}} \delta K + v N \quad \text{subject to:} \quad A K^{\alpha} N^{\beta} = \bar{Q}$$

The Lagrangian becomes: $\mathscr{L} = \delta K + v N + \lambda (\bar{Q} - A K^{\alpha} N^{\beta})$ . This gives the following first order conditions:

(1) $\frac{\partial \mathscr{L}}{\partial K} = \delta - \lambda \alpha A K^{\alpha-1} N^{\beta} = 0$

(2) $\frac{\partial \mathscr{L}}{\partial N} = v - \lambda \beta A K^{\alpha} N^{\beta-1} = 0$

(3) $\frac{\partial \mathscr{L}}{\partial \lambda} = \bar{Q} - A K^{\alpha} N^{\beta} = 0$

Remark: you also need to differentiate by the Lagrangian multiplier (here $\lambda$ in (3) ) for a complete set of FOCs.

(d) Show that the conditional (quantity constrained) demand functions for the constraint $\bar{Q}$ are $K(\bar{Q}, \delta, v)^C = \left(\frac{\bar{Q}}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{v}{\delta} \frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}$ and $N(\bar{Q}, \delta, v)^C = \left(\frac{\bar{Q}}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{\delta}{v} \frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$ **(not exam relevant)**

Remark: It is difficult to derive an analytical expression for the Lagrangian multiplier, $\lambda$, without deriving the cost function (which is outside the scope of this course). In this problem think of it as as the marginal costs of restricting the production level. Compared to the profit maximizing solution (in (b) where $Q^* = Q(K^*, N^*)$ ), it is the reduction in profit divided by the change in the output, i.e., $\lambda(\bar{Q}, \delta, v) = \frac{\pi(K^*, N^*) - \pi(K^C, N^C)}{|\bar{Q} - Q^*|} \forall Q^* \neq \bar{Q}$ and undetermined if $Q^* = \bar{Q}$ as the value within the constraint is zero so that $\lambda$ can take on any value.

Remark: Dropping the absolute operator in the denominator makes a negative adjusted shadow price for $\bar{Q} < Q^*$ , which indicates that by increasing the constraint (moving it closer to $Q^*$) reduces profit losses. With this formulation of the adjusted shadown price, it changes sign from negative to positive at $Q^* = \bar{Q}$ , which makes it easer to interpret numerical results (see part (f)).

(e) The parameters in the production function: $A = 50$, $\alpha = 0{,}4$ , and $\beta = 0{,}3$ .

Prices: the product price $p = 5$, and the input factor prices are $\delta = 4$ (in percent) and $v = 6$ respectively for $K$ and $N$. Find the profit maximizing input factor uses, $K^*$ and $N^*$, the profit maximizing production quantity, $Q^*$, and the corresponding profits (hint: enter the equations with parameters instead of parameter values, and initialize the parameter values in a separate segment of your spreadsheet or computer program).

**Answer:** Insert the values for the parameters in the solution (b) to get:

$$K(p, \delta, v)^* = \left(\frac{1}{pA}\right)^{\frac{1}{\alpha+\beta-1}} \left(\frac{\alpha}{\delta}\right)^{\frac{\beta-1}{\alpha+\beta-1}} \left(\frac{v}{\beta}\right)^{\frac{\beta}{\alpha+\beta-1}} = \left(\frac{1}{5 \cdot 50}\right)^{\frac{1}{0.4+0.3-1}} \left(\frac{0.4}{4}\right)^{\frac{0.3-1}{0.4+0.3-1}} \left(\frac{6}{0.3}\right)^{\frac{0.3}{0.4+0.3-1}} = 22\,844$$

$$N(p, \delta, v)^* = \left(\frac{1}{pA}\right)^{\frac{1}{\alpha+\beta-1}} \left(\frac{\beta}{v}\right)^{\frac{\alpha-1}{\alpha+\beta-1}} \left(\frac{\delta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta-1}} = \left(\frac{1}{5 \cdot 50}\right)^{\frac{1}{0.4+0.3-1}} \left(\frac{0.3}{6}\right)^{\frac{0.4-1}{0.4+0.3-1}} \left(\frac{4}{0.4}\right)^{\frac{0.4}{0.4+0.3-1}} = 11\,422$$

$$Q(K^*, N^*) = A K^{\alpha} N^{\beta} = 50 (22\,844)^{0.4} (11\,422)^{0.3} = 45\,688$$

$$\pi(K^*, N^*) = p Q^* - \delta K - v N = 5 \cdot 45\,688 - 4(22\,844) - 6(11\,422) = 68\,532$$

(f) Using the same prices as in (e) the table below shows the input uses, profits, and the adjusted shadow price (the Lagrangian multiplier) for the following quantity restrictions $\bar{Q}$ = 40 000 to 60 000 by intervals of 2 000.

| Scenario | Profits | Quantity produced | Manmade capital | Nature capital | Shadow price |
|---|---|---|---|---|---|
| Profit maximization | 68532 | 45688 | 22844 | 11422 | . |
| Restriction Q = 30000 | 62320 | 30000 | 12526 | 6263 | -0.396 |
| Restriction Q = 32000 | 63852 | 32000 | 13735 | 6868 | -0.342 |
| Restriction Q = 34000 | 65154 | 34000 | 14978 | 7489 | -0.289 |
| Restriction Q = 36000 | 66233 | 36000 | 16252 | 8126 | -0.237 |
| Restriction Q = 38000 | 67098 | 38000 | 17557 | 8779 | -0.186 |
| Restriction Q = 40000 | 67754 | 40000 | 18892 | 9446 | -0.137 |
| Restriction Q = 42000 | 68208 | 42000 | 20256 | 10128 | -0.088 |
| Restriction Q = 44000 | 68464 | 44000 | 21648 | 10824 | -0.040 |
| Restriction Q = 46000 | 68529 | 46000 | 23067 | 11534 | 0.007 |
| Restriction Q = 48000 | 68407 | 48000 | 24513 | 12257 | 0.054 |
| Restriction Q = 50000 | 68103 | 50000 | 25985 | 12993 | 0.099 |
| Restriction Q = 52000 | 67621 | 52000 | 27483 | 13741 | 0.144 |
| Restriction Q = 54000 | 66964 | 54000 | 29005 | 14503 | 0.189 |
| Restriction Q = 56000 | 66137 | 56000 | 30552 | 15276 | 0.232 |
| Restriction Q = 58000 | 65142 | 58000 | 32123 | 16061 | 0.275 |
| Restriction Q = 60000 | 63984 | 60000 | 33717 | 16858 | 0.318 |

Comment on the following

i. Based upon the adjusted shadow price values, what appears to approximately be the optimal production level, *Q*?

   **Answer:** The adjusted shadow price changes sign somewhere around $\bar{Q}$ = 46 000, with a shadow price slightly less different for $\bar{Q}$ = 44 000 than $\bar{Q}$ = 48 000. This suggests the optimal quantity is a bit less than 46 000.

ii. Any other information in the table that can help you to approximately identify the optimal production level *Q* except the first row? If so, what and why?

   **Answer:** The profits column, where maximum profits appear to be around a production quantiy of 46 000. For the same reasons as in (i).

iii. The optimal input use combinations are in a fixed ratio to each other. Explain why that is the case.

   **Answer:** An examination of the formula for the expansion path shows this. In (b) we found that the ratio of the marginal products (LHS) equals the ratio of input prices (RHS): $\frac{\alpha}{\beta}\frac{N}{K}=\frac{\delta}{v}$ . which gives $K=\frac{\alpha}{\beta}\frac{v}{\delta}N=\frac{0.4}{0.3}\frac{6}{4}N=2N$ , i.e., a linear relation.

iv. The profits decline for production levels above 46 000. What is the reason for that?

   **Answer:** The Cobb Douglas production function $AK^{\alpha}N^{\beta}$ has decreasing returns to scale as $\alpha+\beta=0.4+0.3<1$ . Hence, the total marginal product declines with increasing input use. Then at some point the marginal costs along the expansion path (least cost input factor combination) is higher than the product price. In this case at *Q* = 45 688.