ECN 275/375 Environmental and natural resource economics
Exercise set 2 – Eirik's suggested answers

Exercise 2.1 – utility maximization

The consumer has a Cobb-Douglas utility function: $U(x_{1,}x_2)=x_1^a x_2^b$ where x_1 and x_2 are consumer goods. Let the consumer's money income be Y, and that consumer prices are p_1 for x_1, and p_2 for x_2. Assume this consumer spends all of his income Y on the two goods, x_1 and x_2. Moreover, assume that $a > 0$, $b > 0$, and that $a + b \leq 1$.

(a) What is the marginal utility of consumption for the two goods, x_1 and x_2?

Answer: Differentiate the utility function with regard to the two consumer goods. This yields for good 1 and good 2 respectively:

$$U_1{}'=\frac{\partial U(x_{1,}x_2)}{\partial x_1}=a x_1^{(a-1)} x_2^b \text{ and } U_2{}'=\frac{\partial U(x_1,x_2)}{\partial x_2}=b x_1^a x_2^{(b-1)}$$

(b) Show that the marginal utility of either good is positive.

Answer: Both these are positive expressions, implying that the marginal utility of consumption is positive. With the listed parameter restrictions $a > 0$, and $b > 0$, it follows that both expressions are positive because consumption of either good cannot be negative. Hence, we take exponents of non-negative numbers which gives a positive number.

(c) Show that this consumer's optimal consumption of the two goods, x_1 and x_2., equals

$$x_1^*=\frac{a}{a+b}\frac{Y}{p_1} \quad \text{and} \quad x_2^*=\frac{b}{a+b}\frac{Y}{p_2} \quad \text{respectively.}$$

Answer: Lagrangian for constrained max problem: $L=U(x_{1,}x_2)+\lambda(y-p_1 x_1-p_2 x_2)$

Differentiate with respect to x_1, x_2, and λ. Set to zero to get

$$\frac{\partial L}{\partial x_1}=U_1{}'(x_{1,}x_2)-\lambda p_1=a x_1^{(a-1)} x_2^b-\lambda p_1=0 \quad \text{(see (a) for the marginal utility)}$$

$$\frac{\partial L}{\partial x_2}=U_2{}'(x_{1,}x_2)-\lambda p_2=b x_1^a x_2^{(b-1)}-\lambda p_2=0$$

$$\frac{\partial L}{\partial \lambda}=Y-p_1 x_1-p_2 x_2=0 \quad \text{(the budget constraint)}$$

Substitute and solve: set the first two equations equal to λ to get rid of λ:

$$\lambda=\frac{a x_1^{(a-1)} x_2^b}{p_1}=\frac{b x_1^a x_2^{(b-1)}}{p_2}$$ and factor out (remove some of the exponents for x_1 and x_2).

Trick: $x_1 x_1^{(a-1)}=x^a$ (do the same for x_2) to get

$$\frac{a}{p_1}x_2=\frac{b}{p_2}x_1$$ (rearrange to get either x_1, or x_2 on one side of equal sign, here x_1) :

$$\frac{a}{p_1}x_2\frac{p_2}{b}=\frac{a}{p_1}\frac{p_2}{b}x_2=x_1$$ (and insert into budget constraint above, rearrange). Solution found.

(d) Explain what your solution in (c) really is.

Answer: The expressions $x_1^* = \frac{a}{a+b}\frac{Y}{p_1}$, and $x_2^* = \frac{b}{a+b}\frac{Y}{p_2}$ are the Marshallian demand functions for the two consumer goods (containing the parameter values + consumer income Y, and prices p_1 and p_2).

(e) Show that consumption levels increase with increasing income, Y, and decline with increasing own prices.

Answer: Increasing in consumer income (only shown for x_1)

$x_1^* = \frac{a}{a+b}\frac{Y}{p_1} = (\frac{a}{a+b}\frac{1}{p_1})Y$ which when differentiated wrt. Y yields an expression with only positive number.

Decreasinig in own prices (only shown for x_1):

$x_1^* = \frac{a}{a+b}\frac{Y}{p_1} = (\frac{aY}{a+b})p_1^{(-1)}$ which when differentiated wrt. p_1 yields $-(\frac{aY}{a+b})p_1^{(-2)} < 0$.

Exercise 2.2 – production with mulitple inputs

A Cobb-Douglas production function with two input factors, labor (L) and capital (K) has the form $Q = A L^{\alpha} K^{\beta}$, where $A > 0$, $\alpha > 0$, $\beta > 0$, and $\alpha + \beta \leq 1$, signaling diminishing factor productivity. p is the product price for Q, w is the hourly wage, and r is the cost of capital. For a Cobb-Douglas function to be strictly positive, all inputs must be greater than zero, i.e., $L > 0$ and $K > 0$.

(a) Write down the profit function.

Answer: $\underset{(L,K)}{max}\ \pi = pQ - wL - rK = pAL^{\alpha}K^{\beta} - wL - rK$

(b) Derive the first order conditions (FOC equations) for profit maximization.

Answer: Partially differentiate with respect to the two choice variables L and K to get:

$\frac{\partial \pi}{\partial L} = pQ_L - w = \alpha\, pAL^{\alpha-1}K^{\beta} - w = 0$ and $\frac{\partial \pi}{\partial K} = pQ_K - r = \beta\, pAL^{\alpha}K^{\beta-1} - r = 0$

(c) Show that the optimal ratio of the input factors equals $\frac{L}{K} = \frac{r}{w}\frac{\alpha}{\beta}$. Explain why this implies a linear expansion path?

Answer: Rewrite the two FOCs from (b) to get $\alpha\, pAL^{\alpha-1}K^{\beta} = w$ and $\beta\, pAL^{\alpha}K^{\beta-1} = r$. Excluding the case of zero production allows taking the ratios of these two equations to get:

$\frac{\alpha\, pAL^{\alpha-1}K^{\beta}}{\beta\, pAL^{\alpha}K^{\beta-1}} = \frac{w}{r}$ which simplifies to $\frac{\alpha K}{\beta L} = \frac{w}{r}$, and then transforms to $\frac{L}{K} = \frac{r}{w}\frac{\alpha}{\beta}$.

This yields a linear expansion path because the fraction on the right hand side is a constant for a given set of input prices, which also means that the ratio L/K is constant for all non-zero values of L and K. This implies that if one for example doubles the use of L, one also needs to double the use of K.

(d) Write down the cost minimization problem for producing $\bar{Q} > 0$

Answer: $\underset{(L,K)}{min}\ C = wL + rK$ subject to $\bar{Q} - AL^{\alpha}K^{\beta} \geq 0$

(e) Solve for the cost minimizing combination of *L* and *K*. How does that relate to the answer in (c), and what is the explanation for this result?

Answer: The Lagrangian becomes $\mathcal{L} = wL + rK + \lambda(\bar{Q} - A L^{\alpha} K^{\beta})$ which we can solve as an equality constraint (no need for a Kuhn-Tucker formulation) given the properties of the production function. This gives the following first order conditions:

(1) $\frac{\partial \mathcal{L}}{\partial L} = w - \lambda \alpha A L^{\alpha-1} K^{\beta} = 0$

(2) $\frac{\partial \mathcal{L}}{\partial K} = r - \lambda \beta A L^{\alpha} K^{\beta-1} = 0$

(3) $\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{Q} - A L^{\alpha} K^{\beta} = 0$

Rearranging (1) and (2) gives $\lambda \alpha A L^{\alpha-1} K^{\beta} = w$ and $\lambda \beta A L^{\alpha} K^{\beta-1} = r$. From (1) and (2) it follows that λ cannot be zero when wages and the interest rate are positive. This allows for the same "trick" of dividing one equation by the other as in (c), which yields

$\frac{\lambda \alpha A L^{\alpha-1} K^{\beta}}{\lambda \beta A L^{\alpha} K^{\beta-1}} = \frac{w}{r}$ which again simplifies to $\frac{\alpha K}{\beta L} = \frac{w}{r}$, and then transforms to $\frac{L}{K} = \frac{r}{w} \frac{\alpha}{\beta}$.

The explanation for this result is as follows. The profit maximizing combination of *L* and *K* must also be a cost minimizing solution. There are two explanations for this: (i) The intuitive (economic) explanation: The profit maximizing solution must also be a cost minimizing solution. If this was not the case, inputs could be reallocated and profits would increase. (ii) The mathematical formal explanation. Replace the quantity constraint $\bar{Q}$ in the cost minimizing problem with the profit maximizing production quantity Q^* and solve the cost minimization problem with Q^* as the production constraint. Note that with Q^* as the quantity constraint, the value of the shadow price, λ, equals the prod*uct* pric*e*, *p*, in the profit maximization problem.

Remark: If we want to solve for the optimal input use for $\bar{Q}$, rewrite the ratio of the optimal ratio of inputs to $L = \frac{r}{w} \frac{\alpha}{\beta} K$, and replace for *L* in (3). (you do not need to do that, tedious math without much insights gained).

(f) Suppose we choose to model the emissions of a pollutant, *M*, as an input. How would you expand your profit maximizing model formulation of the above labor-capital input model? Explain your model formulation.

Answer: The profit function in (a) was: $\max_{(L,K)} \pi = pQ - wL - rK = p A L^{\alpha} K^{\beta} - wL - rK$.

Adding pollution as an input: $\max_{(L,K,M)} \pi = p A L^{\alpha} K^{\beta} M^{\gamma} - wL - rK - tM$, where $0 < \gamma < 1$ (setting $\gamma = 0$ means that pollution disappears from the problem as $M^0 = 1$), and *t* is the price per unit of pollution (the emission tax rate). With *t* equal to zero, a producer with no regard for the environment would not consider the amount of pollution, *M*, generated.

Exercise 2.3 – discounting 1

The formula for net present value is written as $NPV(\delta) = \sum_{t=0}^{T} \frac{1}{\beta^t}(B_t - C_t)$ where $\beta = 1 + \delta > 1$.

Let the duration of the project be infinite, i.e., $T = \infty$

Consider a project where investments = 10 take place in year 0 ($t = 0$) and yearly net benefits are constant in fixed monetary terms and equal to 1 (one) from year one and onwards.

(a) Calculate the net present value of this project for a 2% discount rate.

Answer: The trick here is to remember that the NPV in continuous time for a project that lasts infinitely equals 1/δ when yearly net benefits are the same (i.e., $B_1 = B_2 = \ldots = B_T$. As the two formulas (discrete and continuous time) should give the same answer, one can describe total benefits for an infinite project, i.e., B/δ. With $B = 1$ we get 1/0.02 = 50. Subtracting the initial investment and the fact that there is no benefits the first year, the NPV becomes NPV(income stream infinite project = 1/0,02) – discounted benefits (1) in the first year (year 0) – investment first year (=10) = 39.

Remark: the equality between the continuous and discrete time versions implies that

$$\lim(T=\infty)\,NPV(\delta)=\lim(T=\infty)\sum_{t=0}^{T}\frac{1}{\beta^t}(B-C)=\frac{1}{\delta}(B-C)$$

(b) Calculate the net present value of this project for a 5% discount rate.

Answer: Similarly for δ = 5% (0.05) we get 9.

(c) Which of the net present values is highest, and why (give an economic explanation)

Answer: A higher discount rate implies that society values future net benefits lower. For projects with costs (investments) coming early, and benefits in subsequent years, a higher discount rate therefore leads to a lower NPV.

Exercise 2.4 – discounting 2

In a small municipality the local politicians consider a project to make the municpality look nicer. Projected yearly benefits and costs (in 1000 NOK) for the duration (lifetime) of the project is given by the following table:

	Time (year)					
	0	1	2	3	4	5
Benefits	0	111	112	113	114	115
Costs	400	11	12	13	14	15

(a) What are the benefits of the "look nice" project with a discount rate of 9 %? Should the project be implemented at this discount rate?

Answer: Insert into general formula: $NPV(\delta)=\sum_{t=0}^{5}\beta^{-t}(B_t-C_t)$ where β = (1+ δ) = 1,09:

$$NPV(9\%)=-400(1{,}09)^{-0}+100((1{,}09)^{-1}+(1{,}09)^{-2}+(1{,}09)^{-3}+(1{,}09)^{-4}+(1{,}09)^{-5})=-11035$$

The net present value at 9% is negative → the project should not be implemented.

(b) What are the benefits of the same project with a discount rate of 6 %? Should the project be implemented at this discount rate?

Answer: In the same way β = (1+ δ) = 1,06:

$$NPV(6\%)=-400(1{,}06)^{-0}+100((1{,}06)^{-1}+(1{,}06)^{-2}+(1{,}06)^{-3}+(1{,}06)^{-4}+(1{,}06)^{-5})=2324$$

The net present value at 6% is barely positive → the project could be implemented (if the planners are reasonably certain about their benefit and cost estimates)..

(c) Discount rates of 6 or 9 % appear quite high, in particular given the Norwegian Ministry of Finance guidelines of discount rates between 2% and 4% depending upon the type of project. Comment on the municipality's choice of discount rate(s).

Answer: This is a project at the municipal level that according to some has an aura of luxury - "look nice". As such choosing a high discount rate could be justified. Municipalities rarely set their own discount rate, and rather looks at the interests at which they could borrow money (or the interest earned on deposits). One could also argue that nice surroundings have beneficial effects on those living in the area. Such benefits should not be incorporated in the discount rate, but given an estimated value, i.e., possibly leading to higher benefit estimates in the table above.